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On the existence of financial equilibrium when beliefs are private

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Abstract

We consider a pure exchange financial economy, where agents, possibly asymmetrically informed, face an 'exogenous uncertainty', on the future state of nature, and an 'endogenous uncertainty', on the future price in each random state. Namely, every agent forms private price anticipations on every prospective market, distributed along an idiosyncratic probability law. At a sequential equilibrium, all agents expect the 'true' price as a possible outcome and elect optimal strategies at the first period, which clear on all markets at every time period. We show that, provided the endogenous uncertainty is large enough, a sequential equilibrium exists under standard conditions for all types of financial structures and information signals across agents. This result suggests that standard existence problems of sequential equilibrium models, following Hart (1975), stem from the perfect foresight assumption.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

The traditional approach to sequential financial equilibrium relies on Radner's (1972-1979) classical, but restrictive, assumptions that agents have the so-called 'rational expectations' of private information signals, and 'perfect foresight' of future prices. Along the former assumption, agents are endowed, quoting Radner, with 'a model' of how equilibrium prices are determined and (possibly) infer private information of other agents from comparing actual prices and price expectations with theoretical values at a price revealing equilibrium. Along the latter, agents anticipate with certainty exactly one price for each commodity (or asset) in each prospective state, which turns out to be the true price if that state prevails. Both assumptions presume much of agents' inference capacities. Both assumptions lead to classical cases of inexistence of equilibrium, as shown by Radner (1979), Hart (1975), Momi (2000), Busch-Govindan (2004), among others. Building on our earlier papers, we show hereafter that the relevance and properties of the sequential equilibrium model can jointly be improved, if we drop these standard assumptions.

In a first model [4], dropping rational expectations only, we provided the basic tools, concepts and properties for an arbitrage theory, embedding jointly the symmetric and asymmetric information settings. In this model, we showed in [6], standard existence problems of asymmetric information vanished, namely, a financial equilibrium with nominal assets existed, not only generically - as in Radner's (1979) rational expectations model - but under the very same no-arbitrage condition, with symmetric or asymmetric information, namely under the generalized no-arbitrage condition introduced in [4]. This result was consistent with and (partially) extended Cass' (1984) standard existence theorem to the asymmetric information setting.

In a second model [7], dropping both the rational expectation and perfect foresight hypothesis, we extended the above model of asymmetric information to one with all kinds of assets, where agents could forecast future prices with some private uncertainty. This new model embeds both Cornet-de Boisdeffre's (2002) and (2009), and their main results, as particular application cases, but is infinite dimensional and more general. It introduces the basic tools and properties for an arbitrage theory, when agents have asymmetric information and private idiosyncratic price expectations. These concepts and properties generalize Cornet-de Boisdeffre's (2002-2009). In particular, the infinite model displays what information current prices may reveal about the future, to agents having no clue of how equilibrium prices are determined, hence, being prone to uncertainty between (typically) uncountable forecasts. The model shows that no-arbitrage prices, observed on markets, always convey enough information to free markets from arbitrage, after a finite number of inference steps. Then, agents' updated beliefs are said to be revealed by prices.

Formally, the latter model is a two-period pure exchange economy, where agents, possibly asymmetrically informed, face an exogenous uncertainty, represented by finitely many random states of nature, exchange consumption goods on spot markets, and (nominal or real) securities on financial markets, so as to transfer wealth across periods and states. At the first period, besides the above exogenous uncertainty, agents may face an '*endogenous uncertainty*' on the future price, in each state they expect. That is, consumers have private sets of price anticipations, distributed along idiosyncratic probability laws, called beliefs. This uncertainty on prices is said to be *'endogenous'*, because it may affect and is focussed on the endogenous variables.

The latter model of [7] is dealt with throughout this paper. Its equilibrium notion, or '*correct foresight equilibrium'* (C.F.E.), is reached when all agents anticipate the '*true*' price as a possible outcome, and elect optimal strategies, which clear on all markets, ex post. This equilibrium is, indeed, a *sequential* one, i.e., differs from the *temporary* equilibrium notion, introduced by Hicks (1939) and developed by Grandmont (1977, 1982), Green (1973), Hammond (1983), Balasko (2003), among others, in which agents need not anticipate prices correctly at the outset, and may need revise their plans and beliefs, ex post. Building on the arbitrage theory developed in [7], the purpose of the paper is to study the existence conditions of the C.F.E., and to show this concept may bring a response to classical existence problems, following, not only Radner's (1979) rational expectations equilibrium - which we had shown in [6] already, but also Hart (1975), Momi (2001), Busch-Govindan (2004), in particular. We prove that a C.F.E. exists whenever agents' anticipations embed a so-called '*minimum uncertainty set*', presented hereafter. If required, agents' beliefs at the C.F.E. may be revealed (in the above sense) by the equilibrium price itself.

The paper is organized as follows: we present the model, in Section 2, the minimum uncertainty set and existence Theorems, in Section 3, and the Theorem's proof, in Section 4. An Appendix proves technical Lemmas.

2 The basic model

In this Section, we recall the framework and results of the model we had introduced in [7], to which we refer the reader for more details. This model describes a pure-exchange economy with two periods ($t \in \{0, 1\}$), a commodity market and a financial market. At $t = 0$, agents may be asymmetrically informed and uncertain of future prices; and they are also uncertain of the state of nature, which will randomly prevail tomorrow. The sets of agents, $I := \{1, ..., m\}$, commodities, $\mathcal{L} := \{1, ..., L\}$, states of nature, S, and financial assets, $\mathcal{J} := \{1, ..., J\}$, are all finite.

2.1 The model's notations

Throughout, we denote by $\cdot$ the scalar product and $\|.\|$ the Euclidean norm on an Euclidean space and by $\mathcal{B}(K)$ the Borel sigma-algebra of a topological space, K. We let $s = 0$ be the non-random state at $t = 0$ and $S' := \{0\} \cup S$. For all set $\Sigma \subset S'$ and tuple $(s, l, x, x', y, y') \in \Sigma \times \mathcal{L} \times \mathbb{R}^{\Sigma} \times \mathbb{R}^{\Sigma} \times \mathbb{R}^{L\Sigma} \times \mathbb{R}^{L\Sigma}$, we denote by:

- $x_s \in \mathbb{R}$, $y_s \in \mathbb{R}^L$ the scalar and vector, indexed by $s \in \Sigma$, of x, y, respectively;
- y_s^l the l^{th} component of $y_s \in \mathbb{R}^L$;
- $x \leqslant x'$ and $y \leqslant y'$ (respectively, $x << x'$ and $y << y'$) the relations $x_s \leqslant x'_s$ and $y_s^l \leqslant y_s'^l$ (resp., $x_s < x'_s$ and $y_s^l < y_s'^l$) for each $(l, s) \in \{1, ..., L\} \times \Sigma$;
- $x < x'$ (resp., $y < y'$) the joint relations $x \leqslant x'$, $x \neq x'$ (resp., $y \leqslant y'$, $y \neq y'$);
- $\mathbb{R}_+^{L\Sigma} = \{x \in \mathbb{R}^{L\Sigma} : x \geqslant 0\}$ and $\mathbb{R}_+^{\Sigma} := \{x \in \mathbb{R}^{\Sigma} : x \geqslant 0\}$,
  $\mathbb{R}_{++}^{L\Sigma} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$ and $\mathbb{R}_{++}^{\Sigma} := \{x \in \mathbb{R}^{\Sigma} : x >> 0\}$,
- $\mathcal{M}_0 := \{(p_0, q) \in \mathbb{R}_+^L \times \mathbb{R}^J : \|p_0\| + \|q\| = 1\}$;
- $\mathcal{M}_s := \{(s, p_s) : p_s \in \mathbb{R}_+^L,\ \|p_s\| = 1\}$, whenever $s \in S$, and $\mathcal{M} := \cup_{s \in S} \mathcal{M}_s$.

2.2 The commodity and asset markets

The L consumption goods, $l \in \mathcal{L}$, may be exchanged by consumers, on the spot markets of both periods. In each state, $s \in S$, an expectation of a spot price, $p \in \mathbb{R}_+^L$, or the spot price, p, in state s itself, are denoted by the pair $\omega_s := (s, p) \in S \times \mathbb{R}_+^L$, and normalized, at little cost, to the above set $\mathcal{M}_s$.

Each agent, $i \in I$, is granted an endowment, $e_i := (e_{is}) \in \mathbb{R}_+^{LS'}$, which secures her the commodity bundle, $e_{i0} \in \mathbb{R}_+^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_+^L$, in each state $s \in S$, if this state prevails at $t = 1$. To simplify notations, we will also denote $e_{i\omega} := e_{is}$, for every triple $(i, s, \omega) \in I \times S' \times \mathcal{M}_s$. Ex post, the generic i^{th} agent's welfare is measured by a continuous utility index, $u_i : \mathbb{R}^{2L}_+ \to \mathbb{R}_+$, over her consumptions at both dates.

The financial market permits limited transfers across periods and states, via J assets, or securities, $j \in \mathcal{J} := \{1, ..., J\}$, which are exchanged at $t = 0$ and pay off, in commodities and/or in units of account, at $t = 1$. For any spot price, or expectation, $\omega \in \mathcal{M}$, the cash payoffs, $v_j(\omega) \in \mathbb{R}$, of all assets, $j \in \{1, ..., J\}$, conditional on the occurence of price ω, define a row vector, $V(\omega) = (v_j(\omega)) \in \mathbb{R}^J$. By definition and from the continuity of the scalar product, the mapping $\omega \in \mathcal{M} \mapsto V(\omega)$ is continuous. The financial structure may be incomplete, namely, the span, $< (V(\omega_s))_{s \in S} > := \{(V(\omega_s) \cdot z)_{s \in S} : z \in \mathbb{R}^J\}$ may have lower rank (for all prices $(\omega_s) \in \Pi_{s \in S} \mathcal{M}_s$) than $\#S$. From above, assets provide no insurance against endogenous uncertainty.

Agents can take unrestrained positions (positive, if purchased; negative, if sold), in each security, which are the components of a portfolio, $z \in \mathbb{R}^J$. Given an asset price, $q \in \mathbb{R}^J$, a portofolio, $z \in \mathbb{R}^J$, is thus a contract, which costs $q \cdot z$ units of account at $t = 0$, and promises to pay $V(\omega) \cdot z$ units tomorrow, for each spot price $\omega \in \mathcal{M}$, if ω obtains. Similarly, we normalize first period prices, $\omega_0 := (p_0, q)$, to the set $\mathcal{M}_0$.

2.3 Information and beliefs

Consistently with [4], each agent receives a private information signal, $S_i \subset S$, during the first period, which informs her that the true state, i.e., that which will prevail at $t = 1$, will be in S_i. Henceforth, the collection, (S_i), of all agents' signals is set as given and we let $\underline{\mathbf{S}} := \cap_{i=1}^m S_i$. Agents are correctly informed, in the sense that no state of $S \setminus \underline{\mathbf{S}}$ will prevail. They form and may update their private anticipations of future spot prices in each state they expect, and these anticipations are distributed along idiosyncratic probability laws, called beliefs. Formally, we recall from [7]:

Definition 1 *For all probability π, on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, and pair $(\omega := (s,p), \varepsilon) \in \mathcal{M} \times \mathbb{R}_{++}$, we let $B(\omega, \varepsilon) := \{(s', p') \in \mathcal{M} : \|p' - p\| + |s' - s| < \varepsilon\}$ and $P(\pi) := \{\omega \in \mathcal{M} : \pi(B(\omega, \varepsilon)) > 0, \forall \varepsilon > 0\}$ be a compact set, whose elements are called anticipations, expectations or forecasts. A probability, π, on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, is called a belief if the relation $P(\pi) \subset S \times \mathbb{R}^L_{++}$ holds (hence, $\varepsilon := \inf_{(l,(s,p_s)) \in \mathcal{L} \times P(\pi)} \ p^l_s > 0$). We denote by $\mathcal{B}$ the set of all beliefs. A belief, $\pi\prime \in \mathcal{B}$, is said to refine $\pi \in \mathcal{B}$, and we denote it by $\pi\prime \leq \pi$, if $P(\pi\prime) \subset P(\pi)$. Two beliefs, $(\pi, \pi\prime) \in \mathcal{B}^2$, are said to be equivalent, and we denote it by $\pi\prime \sim \pi$, if $P(\pi\prime) = P(\pi)$, and we let $\overset{o}{\pi} := \{\overline{\pi} \in \mathcal{B} : \overline{\pi} \sim \pi\}$ be their (common) equivalence class. We denote by $\mathcal{CB} := \{\overset{o}{\overline{\pi}} : \overline{\pi} \in \mathcal{B}\}$ the set of classes, forming a partition, of $\mathcal{B}$, and by $P(\overset{o}{\pi})$ the expectation support of any class $\overset{o}{\pi} \in \mathcal{CB}$, namely, the set of anticipations, $P(\overset{o}{\pi}) := P(\overline{\pi})$, which is common to all beliefs $\overline{\pi} \in \overset{o}{\pi}$, and which characterizes $\overset{o}{\pi}$. We say that a class, $\overset{o}{\pi}\prime \in \mathcal{CB}$, refines $\overset{o}{\pi} \in \mathcal{CB}$, and denote it by $\overset{o}{\pi}\prime \leq \overset{o}{\pi}$, if $P(\overset{o}{\pi}\prime) \subset P(\overset{o}{\pi})$.*

A collection of beliefs, $(\pi_i) \in \mathcal{B}^m$, is called a structure (of beliefs), and we denote it by $(\pi_i) \in \mathcal{SB}$, if the following condition holds:

(a) $\cap_{i=1}^m P(\pi_i) \neq \varnothing$, i.e., the common anticipation set is non-empty.

Similarly, a collection of classes, $(\overset{o}{\pi}_i) \in \mathcal{CB}^m$, is called a class structure (of beliefs), and we denote it by $(\overset{o}{\pi}_i) \in \mathcal{CSB}$, if $\cap_{i=1}^m P(\overset{o}{\pi}_i) \neq \varnothing$.

Let $((\pi_i), (\pi'_i)) \in \mathcal{SB}^2$, $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ and payoff mapping, V, be given. The couples, $[V, (\pi_i)]$ and $[V, (\overset{o}{\pi}_i)]$, are called, respectively, a structure and a class structure (of payoffs and beliefs). The structure (π'_i) is said to refine (π_i), and we denote it by $(\pi'_i) \leq (\pi_i)$, if the relations $\pi'_i \leq \pi_i$ hold for each $i \in I$. The two structures are equivalent, and we denote it by $(\pi_i) \sim (\pi'_i)$, if both relations $(\pi_i) \leq (\pi'_i)$ and $(\pi'_i) \leq (\pi_i)$ hold. A refinement, $(\pi^*_i) \in \mathcal{SB}$, of $(\pi_i) \in \mathcal{SB}$, is said to be self-attainable if the following Condition holds:

(b) $\cap_{i=1}^m P(\pi^*_i) = \cap_{i=1}^m P(\pi_i)$, i.e., the common anticipation set is left unchanged.

The notions of refinement and self-attainable refinement are defined alike on $\mathcal{CSB}$.

Remark 1 We notice that a class of beliefs identifies to a set of anticipations (not yet ordered by a probability distribution, i.e., by a belief) and a class structure identifies to a collection of anticipations, some of which are common to all agents. Though it is not required for the arbitrage theory we present in Section 2, and, hence, not stated as a condition in the above Definition, we will restrict, in the subsequent Sections 3 and 4, agents' beliefs, $(\pi_i) \in \mathcal{SB}$, to be consistent with the information signals they receive during the first period, that is, restrict beliefs to satisfy $P(\pi_i) \cap \mathcal{M}_s = \varnothing$, for every pair $(i, s) \in I \times S \backslash S_i$. This rationality assumption may also be made in the following sub-Section 2.4, but it is not required, formally.

2.4 Consumers' behavior and the notion of equilibrium

In Section 3, we recall from [7] how agents may refine their anticipations from observing markets in this model. Hereafter, we assume that agents implement their decisions after having reached their (final) structure of beliefs, $(\pi_i) \in \mathcal{SB}$, and observed the market price, $\omega_0 := (p_0, q) \in \mathcal{M}_0$, at $t = 0$, which are set as given and referred to throughout. We also assume that, by the time agents trade, markets have eliminated redundant assets, in the sense that no non-zero portfolio, $z \in \mathbb{R}^J \backslash \{0\}$, is left, which yields an agent (say, $i \in I$) no payoff at all, that is, $V(\omega) \cdot z = 0$, for every $\omega \in P(\pi_i)$. Then, the generic i^{th} agent's consumption set,

$$\boxed{X(\pi_i) := \mathcal{C}\ (P'(\pi_i),\ \mathbb{R}^L_+)},$$

is the set of continuous mappings from $P'(\pi_i) := \{0\} \cup P(\pi_i)$ to $\mathbb{R}^L_+$. A consumption, $x \in X(\pi_i)$, is, thus, a mapping, relating $s = 0$ to a (fixed) consumption decision, $x_0 := x_{\omega_0} \in \mathbb{R}^L_+$, at $t = 0$, and, continuously in $\omega \in P(\pi_i)$, every anticipation, $\omega := (s, p) \in P(\pi_i)$, of the spot price to a random consumption decision, $x_\omega \in \mathbb{R}^L_+$, at $t = 1$, which is conditional on the joint occurence of state s and spot price p at $t = 1$.

Each agent $i \in I$ elects and implements a consumption and investment decision, or strategy, $(x,z) \in X(\pi_i) \times \mathbb{R}^J$, that she can afford on markets, given her endowment, $e_i \in \mathbb{R}_+^{LS'}$, and her expectation set, $P(\pi_i)$. This defines her budget set as follows:

$$\boxed{B_i(\omega_0,\pi_i) := \{(x,z) \in X(\pi_i) \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z; \quad p_s \cdot (x_\omega - e_{i\omega}) \leqslant V(\omega) \cdot z, \forall \omega := (s,p_s) \in P(\pi_i)\}}.$$

An allocation, $(x_i) \in X[(\pi_i)] := \Pi_{i=1}^m X(\pi_i)$, is a collection of consumptions across consumers. We define the following set of attainable allocations:

$$\mathcal{A}((\omega_s),(\pi_i)) := \{(x_i) \in X[(\pi_i)] : \textstyle\sum_{i=1}^m (x_{i0} - e_{i0}) = 0, \ \sum_{i=1}^m (x_{i\omega_s} - e_{i\omega_s}) = 0, \ \forall s \in \underline{\mathbf{S}}, \ s.t. \ \omega_s \in \cap_{i=1}^m P(\pi_i)\} \ ,$$

for every price collection, $(\omega_s) := (\omega_s)_{s \in \underline{\mathbf{S}}} \in \Pi_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$. Each agent $i \in I$ has preferences represented by the V.N.M. utility function:

$$u_i^{\pi_i} : x \in X(\pi_i) \mapsto u_i^{\pi_i}(x) := \int_{\omega \in P(\pi_i)} u_i(x_0, x_\omega) d\pi_i(\omega).$$

The generic i^{th} agent elects a strategy, which maximises her utility function in the buget set, i.e., a strategy of the set $B_i^*(\omega_0,\pi_i) := \arg\max_{(x,z) \in B_i(\omega_0,\pi_i)} \ u_i^{\pi_i}(x)$. The above economy is denoted by $\mathcal{E}$. Its equilibrium concept is defined as follows:

Definition 2 *A collection of prices, $(\omega_s) \in \Pi_{s \in \underline{\mathbf{S}}'} \mathcal{M}_s$, beliefs, $(\pi_i) \in \mathcal{SB}$, and strategies, $(x_i, z_i) \in B_i(\omega_0, \pi_i)$, defined for each $i \in I$, is a correct foresight equilibrium (C.F.E.), or a sequential equilibrium (respectively, a temporary equilibrium) of the economy $\mathcal{E}$, if the following Conditions (a)-(b)-(c)-(d) (resp., Conditions (b)-(c)-(d)) hold:*

(a) $\forall s \in \underline{\mathbf{S}}, \ \omega_s \in \cap_{i=1}^m P(\pi_i)$;

(b) $\forall i \in I, \ (x_i, z_i) \in B_i^*(\omega_0,\pi_i) := \arg\max_{(x,z) \in B_i(\omega_0,\pi_i)} \ u_i^{\pi_i}(x)$;

(c) $(x_i) \in \mathcal{A}((\omega_s),(\pi_i))$;

(d) $\sum_{i=1}^m z_i = 0$.

Under above conditions, $(\pi_i) \in \mathcal{SB}$, or $(\omega_s) \in \Pi_{s \in \underline{\mathbf{S}}'} \mathcal{M}_s$, are said to support equilibrium.

2.5 The model's no-arbitrage prices and the information they reveal

We recall from [7] the definitions of arbitrage-free prices, beliefs, and structures.

Definition 3 *Let a class structure of payoffs and beliefs,* $[V, (\overset{o}{\pi}_i)]$, *a class of beliefs,* $\overset{o}{\pi} \in \mathcal{CB}$, *a representative belief,* $\pi \in \overset{o}{\pi}$, *and a price,* $q \in \mathbb{R}^J$, *be given. The couples,* $(V, \overset{o}{\pi})$ *or* (V, π), *are said to to be* q*-arbitrage-free (hence, arbitrage-free), or* q *to be a no-arbitrage price of* $(V, \overset{o}{\pi})$, *or* (V, π), *if the following equivalent Conditions hold:*

(a) *there is no portfolio* $z \in \mathbb{R}^J$, *such that* $-q \cdot z \geqslant 0$ *and* $V(\omega) \cdot z \geqslant 0$ *for every* $\omega \in P(\pi) = P(\overset{o}{\pi})$, *with at least one strict inequality;*

(b) *there exists a continuous mapping* $\lambda : P(\pi) \to \mathbb{R}_{++}$, *such that* $q = \int_{\omega \in P(\pi)} \lambda(\omega) V(\omega) d\pi(\omega)$.

We let $Q[V, \overset{o}{\pi}]$ *be the set of no-arbitrage prices of* $(V, \overset{o}{\pi})$ *(or* (V, π)*) and* $Q_c[V, (\overset{o}{\pi}_i)] := \cap_{i=1}^m Q[V, \overset{o}{\pi}_i]$ *be the set of common no-arbitrage prices of* $[V, (\overset{o}{\pi}_i)]$. *The class structure* $[V, (\overset{o}{\pi}_i)]$ *is said to be arbitrage-free (resp.,* q*-arbitrage-free) if* $Q_c[V, (\overset{o}{\pi}_i)] \neq \varnothing$ *(resp., if* $q \in Q_c[V, (\overset{o}{\pi}_i)]$*). We say that* q *is a no-arbitrage price (resp., a self-attainable no-arbitrage price) of* $[V, (\overset{o}{\pi}_i)]$ *if there exists a refinement (resp., a self-attainable refinement),* $(\overset{o}{\pi}{}_i^*) \leq (\overset{o}{\pi}_i)$, *such that* $q \in Q_c[V, (\overset{o}{\pi}{}_i^*)]$, *and we denote their set by* $Q[V, (\overset{o}{\pi}_i)]$, *which is non-empty. All above definitions and notations can be stated, equivalently, in terms of any representative structure,* $(\pi_i) \in \Pi_{i=1}^m \overset{o}{\pi}_i$. *We then refer to* $Q_c[V, (\pi_i)] := Q_c[V, (\overset{o}{\pi}_i)]$ *and* $Q[V, (\pi_i)] := Q[V, (\overset{o}{\pi}_i)]$ *as, respectively, the sets of common no-arbitrage prices, and no-arbitrage prices, of the structure* $[V, (\pi_i)]$. *When no confusion arises, the reference to* V *is dropped in the above definitions and notations.*

We now summarize, into the following Claim 1, the main results proved in [7].

Claim 1 *Let* $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ *and a price,* $q \in \mathbb{R}^J$, *be given. The following assertions hold:*

(i) the class structure $(\overset{o}{\pi}_i)$ *(or a structure,* $(\pi_i) \in \Pi_{i=1}^m \overset{o}{\pi}_i$*) is arbitrage-free if and only if there exist no portfolios* $(z_i) \in (\mathbb{R}^J)^I$, *such that* $\sum_{i=1}^m z_i = 0$ *and* $V(\omega_i) \cdot z_i \geqslant 0$ *for* *every couple* $(i, \omega_i) \in I \times P(\pi_i) = I \times P(\overset{o}{\pi}_i)$, *with at least one strict inequality;*

(ii) *there exists a unique coarsest arbitrage-free refinement of* $(\overset{o}{\pi}_i)$, *denoted by* $\overset{o}{\Pi}[V, (\overset{o}{\pi}_i)]$ *or* $\overset{o}{\Pi}[(\overset{o}{\pi}_i)]$. *It is self-attainable. It is equal to* $(\overset{o}{\pi}_i)$ *if and only if* $(\overset{o}{\pi}_i)$ *is arbitrage-free;*

(iii) *for each* $i \in I$, *there exists a set,* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) \in \varnothing \cup \mathcal{CB}$, *also denoted by* $\overset{o}{\Pi}(\overset{o}{\pi}_i, q)$, *and said to be revealed by price* q, *which is empty, if and only if* $\overset{o}{\pi}_i$ *has no* q*-arbitrage-free refinement, and, otherwise, is the coarsest* q*-arbitrage-free refinement of* $\overset{o}{\pi}_i$*;*

(iv) *along assertion* (iii), *above, and Definition 1, the following relations hold:*

($q \in Q[(\overset{o}{\pi}_i)]$, *i.e.,* q *is a no-arbitrage price)* $\Leftrightarrow$ *(* $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \in \mathcal{CSB}$ *)* $\Leftrightarrow$ *(* $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \leq (\overset{o}{\pi}_i)$ *)* $\Leftrightarrow$ *(* $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$ *is the coarsest* q*-arbitrage-free refinement of* $(\overset{o}{\pi}_i)$ *).*

When such conditions hold, the refinement, $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$, *is said to be revealed by price* q. *A refinement is said to be price revealable if it can be revealed by some price.*

(v) $\varnothing \neq \{q \in \mathbb{R}^J : \ \overset{o}{\Pi}[(\overset{o}{\pi}_i)] = (\overset{o}{\Pi}(\overset{o}{\pi}_i, q))\} = Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]]$, *that is, the coarsest refinement,* $\overset{o}{\Pi}[(\overset{o}{\pi}_i)]$, *is revealed by any of its common no-arbitrage prices;*

(vi) $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \leq (\overset{o}{\pi}_i)$ *is self-attainable if and only if* $q \in Q[(\overset{o}{\pi}_i)]$ *is self-attainable.*

We now introduce and discuss a notion of minimum uncertainty on expectations, when agents form their anticipations privately, and state our existence Theorem.

3 An uncertainty principle and the existence of equilibrium

3.1 The existence Theorem

When anticipations are private, there exists a set of minimum uncertainty on future prices, any element of which is an equilibrium price for some structure.

Definition 4 *Let an economy,* $\mathcal{E}$, *as described in Section 2, be given and let* Ω *be its set of sequential equilibria (i.e., of C.F.E.). The minimum uncertainty set,* Δ, *of this economy is the subset of* $\cup_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$, *whose elements support a C.F.E., that is:*

$\Delta = \{\omega_{\underline{s}} \in \cup_{s \in \underline{\mathbf{S}}} \mathcal{M}_s : \exists ((\omega_s^*), (\pi_i^*), [(x_i^*, z_i^*)]) \in \Omega, \ \omega_{\underline{s}} = \omega_{\underline{s}}^* \in \mathcal{M}_{\underline{s}}\}$.

In Section 4, we prove that Δ is non-empty and that a continuum of equilibria exists in standard conditions. Given this, the set Δ is typically uncountable. In the economy, $\mathcal{E}$, agents' information and beliefs are private. So, each agent sees other agents' beliefs as (possibly arbitrary) elements of the set $\mathcal{B}$. We retain the standard small consumer, price-taker, hypothesis, which states that no single agent's belief, or strategy, may have a significative impact on equilibrium prices. The economy, $\mathcal{E}$, is said to be standard if, in addition to that hypothesis, it meets the Conditions:

- ***Assumption A1****:* $\forall i \in I,\ e_i >> 0$;
- ***Assumption A2****:* $\forall i \in I,\ u_i$ *is class* C^1*, strictly concave, strictly increasing.*

Along Theorem 1, in any standard economy, $\mathcal{E}$, the minimum uncertainty set, Δ, is non-empty and a C.F.E. exists, if each agent's forecasts embed that set, Δ.

Theorem 1 *Let a standard economy,* $\mathcal{E}$*, and a class structure,* $[V, (\overset{o}{\pi}_i)]$*, of payoffs and beliefs be given. Let* Δ *be the minimum uncertainty set of the economy* $\mathcal{E}$*, along Definition 4. Then, the following Assertions hold:*

(i) $\exists \varepsilon > 0 : \forall p := (s, \rho) \in \Delta,\ \forall l \in \mathcal{L},\ \rho^l \geqslant \varepsilon$*;*

(ii) $\Delta \neq \varnothing$*.*

Consistently, with the above Assertions (i)*-*(ii)*, if the relation* $\Delta \subset \cap_{i=1}^{m} P(\overset{o}{\pi}_i)$ *holds, that is, if every agent's anticipations embed* Δ*, then, the following Assertions hold:*

(iii) if the class structure, $[V, (\overset{o}{\pi}_i)]$*, is arbitrage-free, every representative structure of beliefs,* $(\pi_i) \in \Pi_{i=1}^{m}\ \overset{o}{\pi}_i$*, supports a C.F.E.;*

(iv) if a self-attainable refinement, $(\overset{o}{\pi}{}_i^*)$*, of* $(\overset{o}{\pi}_i)$*, is arbitrage-free (and such a refinement exists), every representative refinement,* $(\pi_i^*) \in \Pi_{i=1}^{m}\ \overset{o}{\pi}{}_i^*$*, supports a C.F.E.*

(v) if $(\overset{o}{\pi}{}_i^*) \in \mathcal{CSB}$ *is a self-attainable price-revealable refinement of* $(\overset{o}{\pi}_i)$ *(and such a refinement exists), then, every representative refinement,* $(\pi_i^*) \in \Pi_{i=1}^{m}\ \overset{o}{\pi}{}_i^*$*, supports a C.F.E., which is price-revealed, i.e., revealed by the equilibrium price itself.*

Remark 3 Assertion (iv) of Theorem 1 is a direct Corollary of assertion (iii), from replacing the structure $[V, (\overset{o}{\pi}_i)]$ by the arbitrage-free structure $[V, (\overset{o}{\pi}{}_i^*)]$. We let the reader check, as standard from Assumption *A2*, that if $(\pi_i^*) \in \mathcal{SB}$ and $\omega_0 := (p_0, q) \in \mathcal{M}_0$ support a C.F.E., then, $q \in Q_c[V, (\pi_i^*)]$. Consequently, if that supporting structure, $(\pi_i^*) \in \mathcal{SB}$, is price-revealable, then, it is revealed by the equilibrium price, i.e., the C.F.E. is price-revealed. Given this, Assertion (v) of Theorem 1 is a Corollary of Assertion (iv) and only Assertions (i)-(ii)-(iii) need be proved.

Before discussing the Theorem' Condition, $\Delta \subset \cap_{i=1}^m P(\overset{o}{\pi}_i)$, we prove Assertion (i).

Proof of Assertion (i) Let Ω and Δ be the sets of Definition 4, and $\underline{s} \in \underline{\mathbf{S}}$ and $p := (\underline{s}, \rho) \in \Delta$ be given. From the definition, there exist prices, $(\omega_s) \in \Pi_{s \in \underline{\mathbf{S}}'} \mathcal{M}_s$, beliefs, $(\pi_i) \in \mathcal{SB}$, and strategies, $[(x_i, z_i)] \in \Pi_{i=1}^m \; B_i(\omega_0, \pi_i)$, such that $\mathcal{C} := ((\omega_s), (\pi_i), [(x_i, z_i)]) \in \Omega$ and $p = \omega_{\underline{s}}$. Let $e := \max_{(s,l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l > 0$ and $\beta := \sup \frac{\partial u_i}{\partial y^{l'}}(x, y) / \frac{\partial u_i}{\partial y^l}(x, y)$, for $(i, (x, y), (l, l')) \in I \times [0, e]^{2L} \times \mathcal{L}^2$, be given. Then, for each $s \in \underline{\mathbf{S}}'$, the relations $(x_{is}) \geqslant 0$ and $\sum_{i=1}^m (x_{is} - e_{is}) = 0$ hold, from Condition (c) of Definition 2 (applied to $\mathcal{C}$), and imply $x_{is} \in [0, e]^L$, for each $i \in I$. Moreover, the relations $\beta \in \mathbb{R}_{++}$ and $p \in S \times \mathbb{R}_{++}^L$ are standard from Assumption *A2* and Condition (b) of Definition 2 (on $\mathcal{C}$). Let $(l, l') \in \mathcal{L}^2$ be given. We show that $\frac{\rho^l}{\rho^{l'}} \leqslant \beta$. Otherwise, it is standard, from Assumptions *A1*-*A2* and Conditions (b)-(c) of Definition 2 (applied to $\mathcal{C}$), that there exist $i \in I$ and $x \in X(\pi_i)$, identical to x_i in every image, x_ω (for $\omega \in \{0\} \cup P(\pi_i)$), but two, $x_p^{l'} := x_{ip}^{l'} + \frac{\delta}{\rho^{l'}}$ and $x_p^l := x_{ip}^l - \frac{\delta}{\rho^l}$ (for $\delta \in \mathbb{R}_{++}$ small enough), such that $(x, z_i) \in B_i(\omega_0, \pi_i)$ and $u_i^{\pi_i}(x) > u_i^{\pi_i}(x_i)$, which contradicts the fact that $\mathcal{C}$ meets the Condition (b) of Definition 2. We let the reader check the joint relations $\rho >> 0$, $\|\rho\| = 1$ and $\frac{\rho^l}{\rho^{l'}} \leqslant \beta$, which hold, from above, for each $(l, l') \in \mathcal{L}^2$, imply $\rho^l \geqslant \varepsilon := \frac{1}{\beta\sqrt{L}}$, for every $l \in \mathcal{L}$. □

Referring the reader to [7] for the model's refinement of information process, we now discuss the Theorem's condition, namely, the uncertainty principle it embeds.

3.2 Discussing the Theorem's Condition

The Theorem's Condition, $\Delta \subset \cap_{i=1}^{m} P(\overset{o}{\pi}_i)$, is consistent with the model's assumption of price-takers agents seeing other consumers' beliefs as arbitrary: then, Δ is seen as the set of all possible equilibrium prices. It is also consistent with the no-arbitrage condition with asymmetric beliefs, i.e., with anticipations precluding arbitrage (see below). From Theorem 1, the Condition is sufficient for the existence of a C.F.E. It may also be a necessary condition, if beliefs are unpredictable and erratic enough to let any price in Δ become a possible outcome. This situation may arise in times of enhanced uncertainty and volatility. Then, the equilibrium is, typically, non-cooperative. Contrarily, if agents co-operate and share refined beliefs, they might reach a finest sequential equilibrium, which exists from Theorem 2.

Theorem 2 *Let a standard economy, $\mathcal{E}$, and a class structure of payoffs and beliefs, $[V, (\overset{o}{\pi}_i)]$, be given, such that $\Delta \subset \cap_{i=1}^{m} P(\overset{o}{\pi}_i)$. Then, there exists a finest symmetric refinement, $(\overset{o}{\pi}{}_i^{*}) \leq (\overset{o}{\pi}_i)$, such that:*

(i) every representative refinement, $(\pi_i^*) \in \Pi_{i=1}^{m} \overset{o}{\pi}{}_i^{*}$, supports a C.F.E.

Proof We assume that Theorem 1 holds and, non restrictively, that $[V, (\overset{o}{\pi}_i)]$, is such that $P(\overset{o}{\pi}_i) = \Delta$, for each $i \in I$ (hence, arbitrage-free). Let $\mathcal{R}_{(\overset{o}{\pi}_i)} \neq \varnothing$ be the set of symmetric refinements of $(\overset{o}{\pi}_i)$, which satisfy Condition (i) of Theorem 2. Let $\{(\overset{o}{\pi}{}_i^{k})\}_{k \in K} \in \mathcal{R}^K_{(\overset{o}{\pi}_i)}$ be a chain, $\Delta^k := P(\overset{o}{\pi}{}_i^{k})$, for every $(i, k) \in I \times K$, and $\Delta^* := \cap_{k \in K} \Delta^k$. As an intersection of a chain of compact sets, Δ^* is a compact set, hence, non-empty. Let $(\overset{o}{\pi}\prime_i) \leq (\overset{o}{\pi}_i)$ be the symmetric refinement defined by $P(\overset{o}{\pi}\prime_i) = \Delta^*$, for every $i \in I$. We let the reader check by contraposition (as tedious and too long here), from similar arguments as for Lemmas 3-4 (see the Appendix), that $(\overset{o}{\pi}\prime_i) \in \mathcal{R}_{(\overset{o}{\pi}_i)}$, whereas $(\overset{o}{\pi}\prime_i) \leq (\overset{o}{\pi}{}_i^{k})$ holds, for every $k \in K$, by construction. Then, from Zorn's Lemma, $\mathcal{R}_{(\overset{o}{\pi}_i)}$ admits a minimal element, that is, the desired finest refinement of Theorem 2. $\square$

An emprical justification of the Theorem's Condition could be that *relative* prices may be observed and analysed, on long time series, for virtually all types of uncertainty, beliefs, price volatility, speculation, or stability, and past states of nature. From the joint set of relative prices so observed, the set, Δ, of all normalized equilibrium prices, or a bigger set, might be inferred. Below, we hint at another possible explanation, which is theoretical and could be addressed in a subsequent paper.

It is important to recall that agents have no price model in the economy $\mathcal{E}$. Their anticipations start from an initial class structure, $(\overset{o}{\pi}_i) \in \mathcal{CSB}$, which may be updated to a refinement, $(\overset{o}{\pi}{}^*_i) \leq (\overset{o}{\pi}_i)$, when they trade. With no price model, agents can only infer from markets the information that arbitrage opportunities reveal. As shown in [7], from assets' payoffs, such unsophisticated agents may always refine their anticipations, in finitely many steps, up to a coarse, self-attainable, arbitrage-free refinement, $(\overset{o}{\pi}{}^*_i) \leq (\overset{o}{\pi}_i)$, which (typically) is consistent with uncountably many beliefs (those cannot be inferred). If $(\overset{o}{\pi}_i)$, is arbitrage-free, anticipations are kept unchanged, that is, $(\overset{o}{\pi}{}^*_i) = (\overset{o}{\pi}_i)$. Our conjecture is that a similar refinement process as that of [7], through trade, exists on spot markets, in which Δ is a fixed set of no-arbitrage prices, that cannot be ruled out. Since arbitrage is limited by finite endowments, agents may also forecast positive prices outside Δ at equilibrium.

Once inferred from arbitrage as above, the refinement, $(\overset{o}{\pi}{}^*_i) \leq (\overset{o}{\pi}_i)$, cannot be narrowed down from observing prices at $t = 0$. Indeed, along Theorem 1, all possible equilibrium prices at $t = 0$ are supported by refinements, $(\pi^*_i) \in \Pi_{i=1}^m \overset{o}{\pi}{}^*_i$, having the same expectation supports, $P(\pi^*_i) = P(\overset{o}{\pi}{}^*_i)$, for each $i \in I$. So, spot prices at $t = 0$ are non-informative. This outcome is consistent with agents having no price model.

The model leaves room for individual differences, but is also consistent with agents sharing information (along, e.g., Theorem 2). In all cases, agents' final beliefs, say $(\pi_i^*) \in \mathcal{SB}$, belong to same classes of beliefs $\left((\pi_i^*) \in \Pi_{i=1}^m \overset{o}{\pi}_i^*\right)$, but remain private. This privacy, agents' fixed expectations sets, $(P(\overset{o}{\pi}_i^*))$, and the Theorem's Condition restore existence. Indeed, there can be no fall in rank problem a la Hart (1975). The generic i^{th} agent's budget set and strategy are defined *ex ante*, with reference to ex ante conditions, and a fixed expectation set, $P(\overset{o}{\pi}_i^*)$. So, only her *ex ante span* of payoffs matters, $< V, \overset{o}{\pi}_i^* > := \{\omega \in P(\overset{o}{\pi}_i^*) \mapsto V(\omega) \cdot z : z \in \mathbb{R}^J\}$, which is fixed independently of any (equilibrium) price, $p \in \Delta \subset P(\overset{o}{\pi}_i^*)$, to the difference of Hart.

4 The existence proof

Throughout, we set as given a standard economy, $\mathcal{E}$, an arbitrage-free class structure, $[V, (\overset{o}{\pi}_i)]$, a supporting structure of beliefs, $(\pi_i) \in \Pi_{i=1}^m \overset{o}{\pi}_i$, expectation sets, $(P_i) := (P(\overset{o}{\pi}_i))_{i \in I}$. Along Remark 3, we need only prove assertions (ii) and (iii) of Theorem 1. The proof's principle is to construct a sequence of auxiliary economies, with finite expectation sets, refining and tending to the initial expectations sets, $(P_i)_{i \in I}$. Each finite economy admits an equilibrium along Theorem 1 of [6], which we set as given. Then, from the sequence of finite dimensional equilibria, we derive an equilibrium of the initial economy, $\mathcal{E}$. To that aim, we introduce auxiliary sets.

4.1 Auxiliary sets

We divide P_i, for each $i \in I$, in ever finer partitions, and let for each $n \in \mathbb{N}$:

$K_n := \{k_n := (k_n^1, ..., k_n^L) \in (\mathbb{N} \cap [0, 2^n - 1])^L\}$;

$P_{(i,s,k_n)} := P_i \cap (\{s\} \times \Pi_{l \in \{1,...,L\}}]\frac{k_n^l}{2^n}, \frac{k_n^l + 1}{2^n}])$, for every $(s, k_n := (k_n^1, ..., k_n^L)) \in S_i \times K_n$.

For each $(i, s, n, k_n) \in I \times S_i \times \mathbb{N} \times K_n$, such that $P_{(i,s,k_n)} \neq \varnothing$, we select $g^n_{(i,s,k_n)} \in P_{(i,s,k_n)}$ uniquely, and define a set, $G_i^n := \{g^n_{(i,s',k'_n)} : s' \in S_i,\ k'_n \in K_n,\ P_{(i,s',k'_n)} \neq \varnothing\}$, as follows:

- for $n = 0$, we select one $g^0_{(i,s,0)} \in P_{(i,s,0)}$, for all $s \in S_i$, and let $G^0_i := \{g^0_{(i,s,0)} : s \in S_i\}$;

- for $n \in \mathbb{N}^*$ arbitrary, given $G^{n-1}_i := \{g^{n-1}_{(i,s,k_{n-1})} \in P_{(i,s,k_{n-1})} : s \in S_i,\ k_{n-1} \in K_{n-1}\}$, we let, for every $(s, k_n) \in S_i \times K_n$, such that $P_{(i,s,k_n)} \neq \varnothing$,

$$g^n_{(i,s,k_n)} \quad \begin{cases} \textit{be equal to } g^{n-1}_{(i,s,k_{n-1})}, \textit{ if there exists } (k_{n-1}, g^{n-1}_{(i,s,k_{n-1})}) \in K_{n-1} \times G^{n-1}_i \cap P_{(i,s,k_n)} \\ \textit{be set fixed in } P_{(i,s,k_n)}, \textit{ if } \ \in G^{n-1}_i \cap P_{(i,s,k_n)} = \varnothing \end{cases}$$ [2]

This yields a set, $G^n_i := \{g^n_{(i,s,k_n)} \in P_{(i,s,k_n)} : s \in S_i,\ k_n \in K_n\}$, and, by induction, a non-decreasing dense sequence, $\{G^n_i\}_{n \in \mathbb{N}}$, of subsets of P_i, with a good property:

Lemma 1 *There exists $N \in \mathbb{N}^*$, such that the following Assertion holds:*

(i) $\forall(\overset{o}{\pi}{}^*_i) \leq (\overset{o}{\pi}_i),\ (G^N_i \subset P(\overset{o}{\pi}{}^*_i),\ \forall i \in I) \Rightarrow (Q_c[V, (\overset{o}{\pi}{}^*_i)] \neq \varnothing)$.

Proof see the Appendix. □

For every integer $n \geqslant N$ along Lemma 1, and any element $\eta \in]0, 1]$, hereafter set as given, we consider an auxiliary economy, $\mathcal{E}^n_\eta$, which admits an equilibrium, $\mathcal{C}^n$.

4.2 Auxiliary economies, $\mathcal{E}^n_\eta$

We define $\mathbb{N}_N := \mathbb{N} \setminus \{0, 1, ..., N-1\}$, along Lemma 1, and set as given, for each $s \in \underline{\mathbf{S}}$, an arbitrary spot price, $\omega^{N-1}_s := (s, p^{N-1}_s) \in \mathcal{M}_s$. Then, we define, by induction on $n \in \mathbb{N}_N$, a sequence of prices, $\{(\omega^n_s)\} \in (\Pi_{s \in \underline{\mathbf{S}}}\ \mathcal{M}_s)^{\mathbb{N}_N}$, which are, for each $n \in \mathbb{N}_N$, the second period equilibrium prices of the economy $\mathcal{E}^n_\eta$, presented hereafter.

We now let $n \in \mathbb{N}_N$ be given and derive from the set, G^n_i, of sub-Section 4.1, and prices, $(\omega^{n-1}_s) \in \Pi_{s \in \underline{\mathbf{S}}}\ \mathcal{M}_s$, assumed to be defined at the last induction step, an auxiliary economy, $\mathcal{E}^n_\eta$, referred to as the (n, η)-economy, which is of the type described in [6]. Namely, it is a pure exchange economy, with two period ($t \in \{0, 1\}$),

[2] Non restrictively (up to a shift in the upper boundary of $P_{(i,s,k_n)}$), we assume that each $g^n_{(i,s,k_n)} \in P_{(i,s,k_n)}$ is in the interior of $P_{(i,s,k_n)} \neq \varnothing$, to insure that $\pi_i(P_{(i,s,k_n)}) > 0$.

m agents, having incomplete information, and exchanging L goods and J nominal assets, under uncertainty (at $t = 0$) about which state of a finite state space, S^n, will prevail at $t = 1$. Referring to [6], and to the above notations and definitions in the economy $\mathcal{E}$, the generic (n, η)-economy's characteristics are as follows:

- The information structure is the collection, (S_i^n), of sets $S_i^n := \underline{\mathbf{S}} \cup \widetilde{S}_i^n$ (and we let $S_i'^n := \underline{\mathbf{S}}' \cup \widetilde{S}_i^n$), such that $\widetilde{S}_i^n := \{i\} \times G_i^n$ is defined for each $i \in I$. The pooled information set (of the states which may prevail at $t = 1$) is, hence, $\underline{\mathbf{S}} = \cap_{i \in I} S_i^n$. For each $i \in I$, the set $\widetilde{S}_i^n := \{i\} \times G_i^n$ consists of purely formal states, none of which will prevail. The state space of the (n, η)-economy is $S^n = \cup_{i \in I} S_i^n$.
- The $S^n \times J$ payoff matrix, $V^n := (V^n(s^n))$, is defined, with reference to the payoff mapping, V, of the economy $\mathcal{E}$, by $V^n(s) := V((s, p_s^{n-1}))$, for each $s \in \underline{\mathbf{S}}$, and $V^n(s^n) := V(\omega)$, for each $s^n := (i, \omega) \in S^n$. The payoff matrix V^n is purely nominal.
- In each formal state, $s^n := (i, (s, p_s)) \in \widetilde{S}_i^n$, the generic agent $i \in I$ is certain that price $p_s \in \mathbb{R}_{++}^L$, and only that price, can prevail on the s^n-spot market.
- In each realizable state, $s \in \underline{\mathbf{S}}$, the generic agent $i \in I$ has perfect foresight, i.e., anticipates with certainty the true price, say $p_s^n \in \mathbb{R}_{++}^L$ (or $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$).
- The generic i^{th} agent's endowment, $e_i^n := (e_{is^n}^n) \in \mathbb{R}_{++}^{LS_i'^n}$, is defined (with reference to e_i in $\mathcal{E}$) by $e_{is}^n := e_{is}$, for each $s \in \underline{\mathbf{S}}'$, and $e_{is^n}^n := e_{is}$, for each $s^n := (i, (s, p_s)) \in \widetilde{S}_i^n$.
- For every collection of the true market prices, $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$, at $t = 0$, and $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$, for all $s \in \underline{\mathbf{S}}$, at $t = 1$, the generic i^{th} agent has the following consumption set, X_i^n, budget set, $B_i^n(S_i^n, (\omega_s^n))$, and utility function, u_i^n:

$$X_i^n := \mathbb{R}_+^{LS_i'^n};$$

$$B_i^n(S_i^n, (\omega_s^n)) := \left\{(x, z) \in X_i^n \times \mathbb{R}^J : \left[\begin{array}{l} p_0^n \cdot (x_0 - e_{i0}) \leqslant -q^n \cdot z \quad and \quad p_s^n \cdot (x_s - e_{is}) \leqslant V^n(s) \cdot z, \forall s \in \underline{\mathbf{S}} \\ p_s \cdot (x_{s^n} - e_{is}) \leqslant V^n(s^n) \cdot z, \ \forall s^n := (i, (s, p_s)) \in \widetilde{S}_i^n \end{array}\right.\right\};$$

$u_i^n : x \mapsto \sum_{s^n \in S_i^n} \pi_i^n(s^n) u_i(x_0, x_{s^n}),$

where $\underline{\pi_i^n(s^n) := \pi_i(P_{(i,s,k_n)})}$, for every $(s, k_n, s^n) \in S_i \times K_n \times (\{i\} \times [G_i^n \cap P_{(i,s,k_n)}])$, is the probability of the set $\underline{P_{(i,s,k_n)} \neq \varnothing}$, along the belief π_i, $\underline{\text{and } \pi_i^n(s) := \eta, \text{ for each } s \in \underline{\mathbf{S}}}$.

Theorem 1 of [6] and Lemma 1 above yield Lemma 2.

Lemma 2 *The generic* (n, η)*-economy admits an equilibrium, namely a collection of prices,* $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$*, at* $t = 0$*, and* $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$*, in each state* $s \in \underline{\mathbf{S}}$*, and strategies,* $(x_i^n, z_i^n) \in B_i^n(S_i^n, (\omega_s^n))$*, defined for each* $i \in I$*, such that:*

(i) $\forall i \in I,\ (x_i^n, z_i^n) \in \arg\max_{(x,z) \in B_i^n(S_i^n, (\omega_s^n))}\ u_i^n(x)$;

(ii) $\forall s \in \underline{\mathbf{S}}',\ \sum_{i=1}^m (x_{is}^n - e_{is}) = 0$;

(iii) $\sum_{i=1}^m z_i^n = 0$.

Moreover, the equilibrium prices and allocations satisfy the following Assertions:

(iv) $\forall (n, i, s) \in \mathbb{N}_N \times I \times \underline{\mathbf{S}}',\ x_{is}^n \in [0, e]^L$*, where* $e := \max_{(s,l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l$*;*

(v) $\exists \varepsilon \in]0, 1] : p_s^{nl} \geqslant \varepsilon,\ \forall (n, s, l) \in \mathbb{N}_N \times \underline{\mathbf{S}} \times \mathcal{L}$.

Proof see the Appendix. □

Along Lemma 2, we set as given an equilibrium of the (n, η)-economy, namely:

$$\mathcal{C}^n := (\omega_0^n := (p_0^n, q^n), (\omega_s^n), [(x_i^n, z_i^n)]) \in \mathcal{M}_0 \times \Pi_{s \in \underline{\mathbf{S}}}\ \mathcal{M}_s \times \Pi_{i=1}^m\ B_i^n(S_i^n, (\omega_s^n)),$$

which is always referred to. The equilibrium prices, $(\omega_s^n) \in \Pi_{s \in \underline{\mathbf{S}}}\ \mathcal{M}_s$, permit to pursue the induction and define the $(n{+}1, \eta)$-economy in the same way as above, hence, the auxiliary economies and equilibria at all ranks. These meet the following Lemma.

Lemma 3 *For the above sequence,* $\{\mathcal{C}^n\}$*, of equilibria, it may be assumed to exist:*

(i) $\omega_0^\eta := (p_0^\eta, q^\eta) = \lim_{n \to \infty}\ \omega_0^n \in \mathcal{M}_0$ *and* $\omega_s^\eta := (s, p_s^\eta) = \lim_{n \to \infty}\ \omega_s^n \in \mathcal{M}_s$*, for each* $s \in \underline{\mathbf{S}}$*;*

(ii) $(x_{is}^\eta) := \lim_{n \to \infty}\ (x_{is}^n)_{i \in I} \in \mathbb{R}^{Lm}$*, such that* $\sum_{i \in I} (x_{is}^\eta - e_{is}) = 0$*, for each* $s \in \underline{\mathbf{S}}'$*;*

(iii) $(z_i^\eta) = \lim_{n\to\infty}(z_i^n)_{i\in I} \in \mathbb{R}^{Jm}$*, such that* $\sum_{i=1}^m z_i^\eta = 0$.

Moreover, we define, for each $i \in I$*, the following sets and mappings:*

$G_i^\infty := \cup_{n\in\mathbb{N}} G_i^n = \lim_{n\to\infty} \nearrow G_i^n \subset P_i$;

for each $n \in \mathbb{N}$*, the mapping,* $\omega \in P_i \mapsto \mathsf{arg}_i^n(\omega) \in G_i^n$*, from the relations* $(\omega, \mathsf{arg}_i^n(\omega)) \in P^2_{(i,s,k_n)}$*, which hold (for every* $\omega \in P_i$*) for some* $(s, k_n) \in S_i \times K_n$*;*

from Assertion (i) *and Lemma 2-*(v)*, the belief,* $\pi_i^\eta := \frac{1}{1+\eta\#\underline{\mathbf{S}}}(\pi_i + \eta\sum_{s\in\underline{\mathbf{S}}} \delta_s)$*, where* δ_s *is (for each* $s \in \underline{\mathbf{S}}$*) the Dirac's measure of* ω_s^η*;*

the support of $\pi_i^\eta \in \mathcal{B}$*, denoted by* $P_i^\eta := P(\pi_i^\eta) = P_i \cup \{\omega_s^\eta\}_{s\in\underline{\mathbf{S}}}$*;*

for all $(\omega := (s, p_s), z) \in \mathcal{M}\times\mathbb{R}^J$*, the set* $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s\cdot(x - e_{is}) \leqslant V(\omega)\cdot z\}$.

Then, the following Assertions hold, for each $i \in I$*:*

(iv) $G_i^n \subset G_i^{n+1}$*,* $\forall n \in \mathbb{N}$*,* $\overline{G}_i^\infty = P_i$ *and* $\{\mathsf{arg}_i^n(\omega)\}_{n\in\mathbb{N}}$ *converges to* ω *uniformly on* P_i*;*

(v) $\forall s \in \underline{\mathbf{S}}$*,* $\{x_{is}^\eta\} = \arg\max_{x\in B_i(\omega_s^\eta, z_i^\eta)} u_i(x_{i0}^\eta, x)$*, along Assertion* (ii)*; we let* $x_{i\omega_s^\eta}^\eta := x_{is}^\eta$*;*

(vi) the correspondence $\omega \in P_i^\eta \mapsto \arg\max_{x\in B_i(\omega, z_i^\eta)} u_i(x_{i0}^\eta, x)$ *is a continuous mapping, denoted by* $\omega \mapsto x_{i\omega}^\eta$*. The mapping,* $x_i^\eta : \omega \in \{0\} \cup P_i^\eta \mapsto x_{i\omega}^\eta$*, defined from Assertions* (ii) *and* (v) *and above, is a consumption plan, henceforth referred to as* $x_i^\eta \in X(\pi_i^\eta)$*;*

(vii) $u_i^{\pi_i^\eta}(x_i^\eta) = \frac{1}{1+\eta\#\underline{\mathbf{S}}} \lim_{n\to\infty} u_i^n(x_i^n) \in \mathbb{R}_+$.

Proof see the Appendix. □

4.3 An equilibrium of the initial economy

We now prove Assertion (ii) of Theorem 1, via the following Claim.

Claim 2 *The collection of prices,* $(\omega_s^\eta) = \lim_{n\to\infty}(\omega_s^n)$*, beliefs,* (π_i^η)*, allocation,* (x_i^η)*, and portfolios,* $(z_i^\eta) = \lim_{n\to\infty}(z_i^n)$*, of Lemma 3, is a C.F.E. of the economy* $\mathcal{E}$*.*

Proof Let $\mathcal{C}^\eta := ((\omega_s^\eta), (\pi_i^\eta), [(x_i^\eta, z_i^\eta)])$ be defined from Claim 2 and use the notations of Lemma 3. From Lemma 3-(ii)-(iii)-(v)-(vi), $\mathcal{C}^\eta$ meets Conditions (c)-(d) of the above Definition 2 of equilibrium. From the definition, $\{\omega_s^\eta\}_{s\in\underline{\mathbf{S}}} \subset \cap_{i=1}^m P(\pi_i^\eta)$, so $\mathcal{C}^\eta$ meets

Condition (a) of Definition 2. To prove that $\mathcal{C}^\eta$ is a C.F.E., it suffices to show that $\mathcal{C}^\eta$ satisfies the relation $[(x_i^\eta, z_i^\eta)] \in \Pi_{i=1}^m B_i(\omega_0^\eta, \pi_i^\eta)$ and Condition (b) of Definition 2.

Let $i \in I$ be given. From the definition of $\mathcal{C}^n$, the relations $p_0^n \cdot (x_{i0}^n\text{-}e_{i0}) \leqslant -q^n \cdot z_i^n$ and $p_s^n \cdot (x_{is}^n\text{-}e_{is}) \leqslant V((s, p_s^{n-1})) \cdot z_i^n$ hold, for each $(n, s) \in \mathbb{N}_N \times \underline{\mathbf{S}}$, which yield in the limit (from the continuity of the scalar product): $p_0^\eta \cdot (x_{i0}^\eta\text{-}e_{i0}) \leqslant -q^\eta \cdot z_i^\eta$ and $p_s^\eta \cdot (x_{is}^\eta\text{-}e_{is}) \leqslant V(\omega_s^\eta) \cdot z_i^\eta$, for each $s \in \underline{\mathbf{S}}$. The relations $p_s \cdot (x_{i\omega}^\eta\text{-}e_{is}) \leqslant V(\omega) \cdot z_i^\eta$ hold, for all $(s, \omega := (s, p_s)) \in S_i \times P_i$, from Lemma 3-$(v)$-$(vi)$. This implies, from Lemma 3-(vi): $[(x_i^\eta, z_i^\eta)] \in \Pi_{i=1}^m B_i(\omega_0^\eta, \pi_i^\eta)$.

Assume, by contraposition, that $\mathcal{C}^\eta$ fails to meet Definition 2-(b), then, there exist $i \in I$, $(x, z) \in B_i(\omega_0^\eta, \pi_i^\eta)$ and $\varepsilon \in \mathbb{R}_{++}$, such that:

$$(I) \qquad \varepsilon + u_i^{\pi_i^\eta}(x_i^\eta) < u_i^{\pi_i^\eta}(x).$$

We may assume that there exists $\delta \in \mathbb{R}_{++}$, such that:

$$(II) \qquad x_\omega^l \geqslant \delta, \text{ for every } (\omega, l) \in \{0\} \cup P_i^\eta \times \mathcal{L}.$$

If not, for every $\alpha \in [0, 1]$, we define the strategy $(x^\alpha, z^\alpha) := ((1-\alpha)x + \alpha e_i, (1-\alpha)z)$, which belongs to $B_i(\omega_0^\eta, \pi_i^\eta)$, a convex set. From Assumption *A1*, the strategy (x^α, z^α) meets relations (II) whenever $\alpha > 0$. Moreover, from relation (I) and the uniform continuity of $(\alpha, \omega) \in [0, 1] \times P_i^\eta \mapsto u_i(x_0^\alpha, x_\omega^\alpha)$ on a compact set (which holds from Assumption *A2* and the relation $x \in X(\pi_i^\eta)$), the strategy (x^α, z^α) also meets relation (I), for every $\alpha > 0$, small enough. So, we may assume relations (II).

We let the reader check, as immediate from the relations $(x, z) \in B_i(\omega_0^\eta, \pi_i^\eta)$ and $\pi_i^\eta \in \mathcal{B}$ (and the definition of a belief), from Lemma 3-(i), the relations (I) – (II), Assumption *A2*, and the same continuity arguments as above (and the continuity of the scalar product), that we may also assume there exists $\gamma \in \mathbb{R}_{++}$, such that:

$(III)\quad p_0^\eta{\cdot}(x_0-e_{i0})\leqslant\gamma-q^\eta{\cdot}z$ and $p_s{\cdot}(x_\omega-e_{is})\leqslant\gamma+V(\omega){\cdot}z,\ \forall\omega:=(s,p_s)\in P_i^\eta$.

From (III), the continuity of the scalar product (hence, of $\omega\mapsto V(\omega)$) and Lemma 3-(i)-(iii)-(iv), there exists $N_1\in\mathbb{N}_N$, such that, for every $n\geqslant N_1$:

$$(IV)\quad\left[\begin{array}{l} p_0^n{\cdot}(x_0-e_{i0})\leqslant -q^n{\cdot}z \\ p_s^n{\cdot}(x_{\omega_s^\eta}-e_{is})\leqslant V^n(s){\cdot}z,\ \forall s\in\underline{\mathbf{S}} \\ p_s{\cdot}(x_\omega-e_{is})\leqslant V(\omega){\cdot}z,\ \forall\omega:=(s,p_s)\in G_i^n \end{array}\right. .$$

Along relations (IV) and Lemma 3-(i)-(v)-(vi), for each $n\geqslant N_1$, we let $(x^n,z)\in B_i^n(S_i^n,(\omega_s^n))$ be the strategy defined by $x_0^n:=x_0$, $x_s^n:=x_{\omega_s^\eta}$, for every $s\in\underline{\mathbf{S}}$, and $x_{s^n}^n:=x_\omega$, for every $s^n:=(i,\omega)\in\widetilde{S}_i^n$, and recall that:

- $u_i^{\pi_i^\eta}(x):=\frac{1}{1+\eta\#\underline{\mathbf{S}}}\int_{\omega\in P_i}u_i(x_0,x_\omega)du_i^{\pi_i}(\omega)+\frac{\eta}{1+\eta\#\underline{\mathbf{S}}}\sum_{s\in\underline{\mathbf{S}}}u_i(x_0,x_{\omega_s^\eta})$;
- $u_i^n(x^n):=\sum_{s^n\in S_i^n\setminus\underline{\mathbf{S}}}u_i(x_0,x_{s^n})\pi_i^n(s^n)+\eta\sum_{s\in\underline{\mathbf{S}}}u_i(x_0,x_{\omega_s^\eta})$.

Then, from above, Lemma 3-(i)-(iv) and the uniform continuity of $x\in X(\pi_i^\eta)$ and u_i on compact sets, there exists $N_2\geqslant N_1$ such that (with $\beta:=(1{+}\eta\#\underline{\mathbf{S}})$):

$(V)\qquad |\beta u_i^{\pi_i^\eta}(x)\text{-}u_i^n(x^n)|\leqslant\int_{\omega\in P_i}|u_i(x_0,x_\omega)\text{-}u_i(x_0,x_{\mathsf{arg}_i^n(\omega)})|du_i^{\pi_i}(\omega)<\frac{\varepsilon}{2}$, for every $n\geqslant N_2$.

From equilibrium conditions and Lemma 3-(vii), there exists $N_3\geqslant N_2$, such that:

$(VI)\qquad u_i^n(x^n)\leqslant u_i^n(x_i^n)<\frac{\varepsilon}{2}+\beta u_i^{\pi_i^\eta}(x_i^\eta)$, for every $n\geqslant N_3$ (with $\beta:=(1{+}\eta\#\underline{\mathbf{S}})$).

Let $n\geqslant N_3$ be given. The above Conditions (I)-(V)-(VI) yield, jointly:

$\beta u_i^{\pi_i^\eta}(x)<\frac{\varepsilon}{2}+u_i^n(x^n)<\varepsilon+\beta u_i^{\pi_i^\eta}(x_i^\eta)<\beta u_i^{\pi_i^\eta}(x)$.

This contradiction proves that $\mathcal{C}^\eta$ meets Condition (b) of Definition 2, hence, from above, that $\mathcal{C}^\eta$ is a C.F.E. The sets Ω, of C.F.E., and Δ, of minimum uncertainty, of Defintion 4, which contains $\{\omega_s^\eta\}_{s\in\underline{\mathbf{S}}}$, are non-empty, i.e., Theorem 1-(ii) holds. $\square$

Claim 3, below, completes the proof of Theorem 1 via the following Lemma.

Lemma 4 *For each* $(i,k) \in I{\times}\mathbb{N}$*, we let* $\eta_k := \frac{1}{2^k}$ *, denote simply* $u_i^k := u_i^{\pi_i^{\eta_k}}$ *and by* $\mathcal{C}^k = ((\omega_s^k),(\pi_i^k),[(x_i^k,z_i^k)])$ *the related C.F.E.,* $\mathcal{C}^{\eta_k}$*, of Claim 2, and we define the set,* $B_i(\omega,z) := \{x \in \mathbb{R}_+^L : p_s{\cdot}(x{-}e_{is}) \leqslant V(\omega){\cdot}z\}$*, for all* $(\omega := (s,p_s),z) \in P_i{\times}\mathbb{R}^J$*. Then, whenever* $\Delta{\subset}\cap_{i=1}^m P_i$ *along Definition 4, the following Assertions hold for each* $i \in I$*:*

(i) *for each* $s \in \underline{\mathbf{S}}'$*, it may be assumed to exist prices,* $\omega_s^* = \lim_{k\to\infty}\ \omega_s^k \in \mathcal{M}_s$*, such that* $\{\omega_s^*\}_{s\in\underline{\mathbf{S}}} \subset \cap_{i=1}^m P_i$*, and consumptions,* $x_{is}^* = \lim_{k\to\infty}\ x_{is}^k$*, such that* $\sum_{i\in I}(x_{is}^*{-}e_{is}) = 0$*;*

(ii) *it may be assumed to exist portfolios,* $z_i^* = \lim_{k\to\infty} z_i^k$*, such that* $\sum_{i=1}^m z_i^* = 0$*;*

(iii) $\forall s \in \underline{\mathbf{S}}$*,* $\{x_{is}^*\} = \arg\max_{x\in B_i(\omega_s^*,z_i^*)}\ u_i(x_{i0}^*,x)$ *along Assertion* (i)*; we let* $x_{i\omega_s^*}^* := x_{is}^*$*;*

(iv) *the correspondence* $\omega \in P_i \mapsto \arg\max_{x\in B_i(\omega,z_i^*)}\ u_i(x_{i0}^*,x)$ *is a continuous mapping, denoted by* $\omega \mapsto x_{i\omega}^*$*. The mapping* $x_i^* : \omega \in \{0\}\cup P_i \mapsto x_{i\omega}^*$*, defined from Assertions* (i)*-*(iii) *and above, is a consumption plan, referred to as* $x_i^* \in X(\pi_i)$*;*

(v) *for every* $x \in X(\pi_i)$*,* $u_i^{\pi_i}(x) = \lim_{k\to\infty} u_i^k(x) \in \mathbb{R}_+$ *and* $u_i^{\pi_i}(x_i^*) = \lim_{k\to\infty} u_i^k(x_i^k) \in \mathbb{R}_+$*.*

Proof see the Appendix. □

Claim 3 *Whenever* $\Delta \subset \cap_{i=1}^m P_i$*, the collection of prices,* $(\omega_s^*) = \lim_{k\to\infty}(\omega_s^k)$*, beliefs,* (π_i)*, allocation,* (x_i^*)*, and portfolios,* $(z_i^*) = \lim_{k\to\infty}(z_i^k)$*, of Lemma 4, is a C.F.E.*

Proof The proof is similar to that of Claim 2. We assume that $\Delta \subset \cap_{i=1}^m P_i$ and let $\mathcal{C}^* := ((\omega_s^*),(\pi_i),[(x_i^*,z_i^*)])$ be defined from Lemma 4, whose notations will be used throughout. Given $(i,k) \in I \times \mathbb{N}$, the relations $\{\omega_s^k\}_{s\in\underline{\mathbf{S}}} \subset \Delta \subset \cap_{i=1}^m P_i$ hold from Claim 2, and imply that $P(\pi_i^k) = P_i$, hence, $B_i(\omega_0^*,\pi_i)$ and $B_i(\omega_0^k,\pi_i^k)$ may only differ by one budget constraint at $t = 0$. From Lemma 4, $\mathcal{C}^*$ meets Conditions (a)-(c)-(d) of Definition 2. Moreover, for every $(i,k) \in I \times \mathbb{N}$, the relations $p_0^k{\cdot}(x_{i0}^k\text{-}e_{i0}) \leqslant -q^k{\cdot}z_i^k$ hold, from Claim 2, and, passing to the limit, yield $p_0^*{\cdot}(x_{i0}^*\text{-}e_{i0}) \leqslant -q^*{\cdot}z_i^*$, which implies, from Lemma 4-(iv) and above: $(x_i^*,z_i^*) \in B_i(\omega_0^*,\pi_i)$, for each $i \in I$. Thus, Claim 3 will be proved if we show that $\mathcal{C}^*$ meets Condition (b) of Definition 2. By contraposition, assume this is not the case, i.e., there exists $(i,(x,z),\varepsilon) \in I \times B_i(\omega_0^*,\pi_i) \times \mathbb{R}_{++}$, such that:

$$(I) \qquad \varepsilon + u_i^{\pi_i}(x_i^*) < u_i^{\pi_i}(x).$$

From relation $(x,z) \in B_i(\omega_0^*,\pi_i)$, Lemma 4-$(i)$ and Assumptions *A1-A2*, the relation:

$$(II) \qquad p_0^* \cdot (x_0 \text{-} e_{i0}) \leqslant \gamma - q^* \cdot z, \text{ for some } \gamma \in \mathbb{R}_{++}, \text{ may also be assumed.}$$

From (II), Lemma 4-(i), continuity arguments and the identity of $B_i(\omega_0^*,\pi_i)$ and $B_i(\omega_0^k,\pi_i^k)$ on all second period budget constraints, there exists $K \in \mathbb{N}$, such that:

$$(III) \qquad (x,z) \in B_i(\omega_0^k,\pi_i) = B_i(\omega_0^k,\pi_i^k), \text{ for every } k \geqslant K.$$

Relations (I)-(III), Lemma 4-(v) and the fact that $\mathcal{C}^k$ is a C.F.E., yield:

$$(IV) \qquad u_i^{\pi_i}(x) < \tfrac{\varepsilon}{2} + u_i^k(x) \leqslant \tfrac{\varepsilon}{2} + u_i^k(x_i^k) < \varepsilon + u_i^{\pi_i}(x_i^*) < u_i^{\pi_i}(x), \text{ for } k \geqslant K \text{ big enough.}$$

This contradiction proves that $\mathcal{C}^*$ meets Definition 2-(b), i.e., is a C.F.E. From Claim 2, Remark 3, the arbitrary choice of the arbitrage-free class structure, $[V,(\overset{o}{\pi}_i)]$, and representative structure, $(\pi_i) \in \Pi_{i=1}^m \overset{o}{\pi}_i$, the proof of Theorem 1 is complete. □

Appendix: proof of the Lemmas

Lemma 1 *There exists $N \in \mathbb{N}^*$, such that the following Assertion holds:*

(i) $\forall (\overset{o}{\pi}{}_i^*) \leq (\overset{o}{\pi}_i),\ (G_i^N \subset P(\overset{o}{\pi}{}_i^*),\ \forall i \in I) \Rightarrow (Q_c[V,(\overset{o}{\pi}{}_i^*)] \neq \varnothing)$.

Proof Let the arbitrage-free class structure, $[V,(\overset{o}{\pi}_i)]$, expectation sets, $(P_i) := (P(\overset{o}{\pi}_i))$, and sequences, $\{(G_i^n)\}_{n \in \mathbb{N}}$, be defined as in Section 4. For each $(i,n) \in I \times \mathbb{N}$, we consider the vector space $Z_i^n := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in G_i^n\}$ and its orthogonal, $Z_i^{n\perp}$, and, similarly, $Z_i^* := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i\}$ and $Z_i^{*\perp}$. We show, first, that, for each $i \in I$, there exists $N_i \in \mathbb{N}$, such that $Z_i^n = Z_i^*$, for every $n \geqslant N_i$. Indeed, since $\{G_i^n\}_{n\in\mathbb{N}}$ is non-decreasing, $\{Z_i^n\}_{n\in\mathbb{N}}$ is non increasing in $\mathbb{R}^J$, hence, stationary, that is, there exists $N_i \in \mathbb{N}$, such that $Z_i^n = Z_i^{N_i}$, for every $n \geqslant N_i$. From the definition, $Z_i^* \subset Z_i^{N_i}$. From the fact that $\lim_{n\to\infty} \nearrow G_i^n = \cup_{n\in\mathbb{N}} G_i^n$ is dense in P_i, we easily show, by contraposition, that $Z_i^* = Z_i^{N_i}$ (for all $n > N_i$, take $z_n \in Z_i^{*\perp} \cap Z_i^n$, such that $\|z_n\| = 1$ and derive a contradiction). We let $N^o = \max_{i\in I} N_i$ and define the compact set, $Z := \{(z_i) \in \Pi_{i=1}^m Z_i^{*\perp} : \|(z_i)\| = 1,\ \sum_{i=1}^m\ z_i \in \sum_{i=1}^m\ Z_i^*\}$.

Assume, by contraposition, that Lemma 2 fails. Then, from Claim 1-(i) and above, for every $n \geqslant N^o$, there exist an integer, $N_n \geqslant n$, expectation sets, $(P_i^{N_n})$, such that $G_i^{N_n} \subset P_i^{N_n} \subset P_i$, for each $i \in I$, and portfolios, $(z_i^n) \in Z$, such that $V(\omega_i) \cdot z_i^n \geqslant 0$ holds for every $(i, \omega_i) \in I \times P_i^{N_n}$, with one strict inequality. The sequence, $\{(z_i^n)\}_{n\geqslant\mathbb{N}^o}$, may be assumed to converge in a compact set, say to $(z_i^*) \in Z$. From the continuity of the scalar product and the fact that, for each $i \in I$, $\lim_{n\to\infty} G_i^n = \cup_{n\in\mathbb{N}} G_i^n$ is dense in P_i, the above relations on $\{(z_i^n)\}_{n\geqslant\mathbb{N}^o}$, imply, in the limit, that $V(\omega_i) \cdot z_i^* \geqslant 0$ holds, for every $(i, \omega_i) \in I \times P_i$, with one strict inequality, since $(z_i^*) \in Z$. This contradicts the fact that $[V, \overset{o}{\pi}_i]$ is arbitrage-free. This contradiction proves Lemma 1. □

Lemma 2 *The generic* (n, η)*-economy admits an equilibrium, namely a collection of prices,* $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$*, at* $t = 0$*, and* $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$*, in each state* $s \in \underline{\mathbf{S}}$*, and strategies,* $(x_i^n, z_i^n) \in B_i^n(S_i^n, (\omega_s^n))$*, defined for each* $i \in I$*, such that:*

(i) $\forall i \in I,\ (x_i^n, z_i^n) \in \arg\max_{(x,z)\in B_i^n(S_i^n,(\omega_s^n))}\ u_i^n(x)$;

(ii) $\forall s \in \underline{\mathbf{S}}',\ \sum_{i=1}^m\ (x_{is}^n - e_{is}) = 0$;

(iii) $\sum_{i=1}^m z_i^n = 0$.

Moreover, the equilibrium prices and allocations satisfy the following Assertions:

(iv) $\forall (n, i, s) \in \mathbb{N}_N \times I \times \underline{\mathbf{S}}',\ x_{is}^n \in [0, e]^L,$ *where* $e := \max_{(s,l)\in S'\times\mathcal{L}} \sum_{i=1}^m e_{is}^l$;

(v) $\exists \varepsilon \in]0, 1] : p_s^{nl} \geqslant \varepsilon,\ \forall (n, s, l) \in \mathbb{N}_N \times \underline{\mathbf{S}} \times \mathcal{L}.$

Proof Let $n \in \mathbb{N}_N$ be given. From Lemma 1 and the fact that $\underline{\mathbf{S}}$ is a set of common states for all agents, the structure $[V^n, (S_i^n)]$ is arbitrage-free, along [6], on a purely financial market. Moreover, the (n, η)-economy is, formally, one of the type presented in [6] and, from above, admits an equilibrium along Definition 3 and Theorem 1 of [6] and its proof, more precisely (up to a slight change in notations), it admits a collection of prices, $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$, and $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$, for each $s \in \underline{\mathbf{S}}$, and strategies, $(x_i^n, z_i^n) \in B_i^n(S_i^n, (\omega_s^n))$, defined for each $i \in I$, which satisfy Assertions (i)-(ii)-(iii) of Lemma 2 (which, hence, hold). The proof of Assertions (iv)-(v) is similar to that of Assertion (i) of Theorem 1, given above, and left to the reader. $\square$

Lemma 3 *For the above sequence, $\{\mathcal{C}^n\}$, of equilibria, it may be assumed to exist:*

(i) $\omega_0^\eta := (p_0^\eta, q^\eta) = \lim_{n\to\infty} \omega_0^n \in \mathcal{M}_0$ *and* $\omega_s^\eta := (s, p_s^\eta) = \lim_{n\to\infty} \omega_s^n \in \mathcal{M}_s$, *for each* $s \in \underline{\mathbf{S}}$;

(ii) $(x_{is}^\eta) := \lim_{n\to\infty} (x_{is}^n)_{i\in I} \in \mathbb{R}^{Lm}$, *such that* $\sum_{i\in I} (x_{is}^\eta - e_{is}) = 0$, *for each* $s \in \underline{\mathbf{S}}'$;

(iii) $(z_i^\eta) = \lim_{n\to\infty} (z_i^n)_{i\in I} \in \mathbb{R}^{Jm}$, *such that* $\sum_{i=1}^m z_i^\eta = 0$.

Moreover, we define, for each $i \in I$, the following sets and mappings:

$G_i^\infty := \cup_{n\in\mathbb{N}} G_i^n = \lim_{n\to\infty} \nearrow G_i^n \subset P_i$;

for each $n \in \mathbb{N}$, the mapping, $\omega \in P_i \mapsto \mathsf{arg}_i^n(\omega) \in G_i^n$, from the relations $(\omega, \mathsf{arg}_i^n(\omega)) \in P^2_{(i,s,k_n)}$, which hold (for every $\omega \in P_i$) for some $(s, k_n) \in S_i \times K_n$;

from Assertion (i) and Lemma 2-(v), the belief, $\pi_i^\eta := \frac{1}{1+\eta \# \underline{\mathbf{S}}}(\pi_i + \eta \sum_{s\in\underline{\mathbf{S}}} \delta_s)$, where δ_s is (for each $s \in \underline{\mathbf{S}}$) the Dirac's measure of ω_s^η;

the support of $\pi_i^\eta \in \mathcal{B}$, denoted by $P_i^\eta := P(\pi_i^\eta) = P_i \cup \{\omega_s^\eta\}_{s\in\underline{\mathbf{S}}}$;

for all $(\omega := (s, p_s), z) \in \mathcal{M} \times \mathbb{R}^J$, the set $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s \cdot (x - e_{is}) \leqslant V(\omega) \cdot z\}$.

Then, the following Assertions hold, for each $i \in I$:

(iv) $G_i^n \subset G_i^{n+1}$, $\forall n \in \mathbb{N}$, $\overline{G}_i^\infty = P_i$ *and* $\{\mathsf{arg}_i^n(\omega)\}_{n\in\mathbb{N}}$ *converges to ω uniformly on P_i;*

(v) $\forall s \in \underline{\mathbf{S}}$, $\{x_{is}^\eta\} = \arg\max_{x\in B_i(\omega_s^\eta, z_i^\eta)} u_i(x_{i0}^\eta, x)$, *along Assertion (ii); we let* $x_{i\omega_s^\eta}^\eta := x_{is}^\eta$;

(vi) *the correspondence* $\omega \in P_i^\eta \mapsto \arg\max_{x\in B_i(\omega, z_i^\eta)} u_i(x_{i0}^\eta, x)$ *is a continuous mapping,* *denoted by* $\omega \mapsto x^\eta_{i\omega}$. *The mapping,* $x^\eta_i : \omega \in \{0\} \cup P^\eta_i \mapsto x^\eta_{i\omega}$, *defined from Assertions* (ii) *and* (v) *and above, is a consumption plan, henceforth referred to as* $x^\eta_i \in X(\pi^\eta_i)$;

(vii) $u^{\pi^\eta_i}_i(x^\eta_i) = \frac{1}{1+\eta \# \underline{\mathbf{S}}} \lim_{n\to\infty} u^n_i(x^n_i) \in \mathbb{R}_+$.

Proof Assertions (i)-(ii) result from Lemma 2-(iv) and compactness arguments. $\square$

Assertion (iii) For all $(i,n) \in I \times \mathbb{N}_N$, we let $Z^*_i := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i\}$ and recall from the proof of Lemma 1 that $Z^*_i = \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in G^n_i\}$. We show that the portfolio sequence $\{(z^n_i)_{i\in I}\}$ is bounded in $\mathbb{R}^{Jm}$. Indeed, let $\delta := \max_{i\in I} \|e_i\|$. The definition of $\{\mathcal{C}^n\}_{n\in\mathbb{N}_N}$ yields, from budget constraints and clearance conditions:

(I) $\qquad [\sum_{i=1}^m z^n_i = 0$ *and* $V(\omega_i) \cdot z^n_i \geqslant -\delta,\ \forall (i,\omega_i) \in I \times G^n_i]$, for every $n \in \mathbb{N}_N$.

Assume, by contradiction, $\{(z^n_i)\}$ is unbounded, i.e., there exists an extracted sequence, $\{(z^{\varphi(n)}_i)\}$, such that $n < \|(z^{\varphi(n)}_i)\| \leqslant n+1$, for all $n \in \mathbb{N}_N$. From (I), the portfolios $(z^{*n}_i) := \frac{1}{n}(z^{\varphi(n)}_i)$ meet, for all $n \in \mathbb{N}_N$, the relations $1 < \|(z^{*n}_i)\| \leqslant 1 + \frac{1}{n}$ and:

(II) $\qquad \sum_{i=1}^m z^n_i = 0$ *and* $V(\omega_i) \cdot z^{*n}_i \geqslant -\frac{\delta}{n},\ \forall (i,\omega_i) \in I \times G^n_i$.

From (II), the density of G^∞_i in P_i, scalar product continuity and above, the sequence $\{(z^{*n}_i)\}$ may be assumed to converge, say to $(\overline{z}_i)$, such that $\|(\overline{z}_i)\| = 1$ and:

(III) $\qquad \sum_{i=1}^m \overline{z}_i = 0$ *and* $V(\omega_i) \cdot \overline{z}_i \geqslant 0,\ \forall (i,\omega_i) \in I \times P_i$.

Relations (III) and the fact that $[V, (\overset{o}{\pi}_i)]$ is arbitrage-free imply, from Claim 1, $(\overline{z}_i) \in \Pi_{i=1}^m Z^*_i$ and, from the elimination of redundant assets (see sub-Section 2.4), $(\overline{z}_i) = 0$, which contradicts the fact that $\|(\overline{z}_i)\| = 1$. Hence, the sequence $\{(z^n_i)\}$ is bounded and may be assumed to converge, say to $(z^\eta_i) \in \mathbb{R}^{Jm}$. Then, the relation $\sum_{i=1}^m z^\eta_i = 0$ results asymptotically from clearance conditions in Lemma 2-(iv). $\square$

Assertions (iv) is immediate from the definitions and compactness arguments. $\square$

Assertion (v) Let $(i,s)\in I\times\underline{\mathbf{S}}$ be given. For every $(n,\omega:=(s,p_s),\omega',z)\in\mathbb{N}_N\times\mathcal{M}_s\times\mathcal{M}_s\times\mathbb{R}^J$, we consider the following (possibly empty) sets:

$B_i(\omega,z):=\{y\in\mathbb{R}^L_+ : p_s\cdot(y-e_{is})\leqslant V(\omega)\cdot z\}$ and $B'_i(\omega,\omega',z):=\{y\in\mathbb{R}^L_+ : p_s\cdot(y-e_{is})\leqslant V(\omega')\cdot z\}$.

For each $n>N$, the fact that $\mathcal{C}^n$ is a (n,η)-equilibrium implies, from Lemma 2:

$$(I)\qquad (\omega_s^{n-1},\omega_s^n)\in\mathcal{M}_s^2 \text{ and } x_{is}^n\in\arg\max_{y\in B'_i(\omega_s^n,\omega_s^{n-1},z_i^n)} u_i(x_{i0}^n,y).$$

As a standard application of Berge's Theorem (see, e.g., [8], p. 19), the correspondence $(x,\omega,\omega',z)\in\mathbb{R}^L_+\times\mathcal{M}_s\times\mathcal{M}_s\times\mathbb{R}^J\mapsto\arg\max_{y\in B'_i(\omega,\omega',z)}\ u_i(x,y)$, which is actually a mapping (from Assumption *A2*), is continuous at $(x_{i0}^\eta,\omega_s^\eta,\omega_s^\eta,z_i^\eta)$, since u_i and B'_i are continuous. Moreover, the relation $(x_{i0}^\eta,x_{is}^\eta,\omega_s^\eta,z_i^\eta)=\lim_{n\to\infty}(x_{i0}^n,x_{is}^n,\omega_s^n,z_i^n)$ holds from Lemma 2-(i)-(ii)-(iii). Hence, the relations (I) pass to that limit and yield:

$$\{x_{i\omega_s^\eta}^\eta\}:=\{x_{is}^\eta\}=\arg\max_{y\in B_i(\omega_s^\eta,z_i^\eta)} u_i(x_{i0}^\eta,y). \qquad \square$$

Assertion (vi) Let $i\in I$ be given. For every $(\omega,n)\in P_i\times\mathbb{N}_N$, the fact that $\mathcal{C}^n$ is a (n,η)-equilibrium and Assumption *A2* yield:

$$(I)\quad \{x_{i\mathsf{arg}_i^n(\omega)}^n\}=\arg\max_{y\in B_i(\mathsf{arg}_i^n(\omega),z_i^n)} u_i(x_{i0}^n,y).$$

From Lemma 2-(ii)-(iii)-(iv), the relation $(\omega,x_{i0}^\eta,z_i^\eta)=\lim_{n\to\infty}(\mathsf{arg}_i^n(\omega),x_{i0}^n,z_i^n)$ holds, whereas, from Assumption *A2* and ([8], p. 19), the correspondence $(x,\omega,z)\in\mathbb{R}^L_+\times P_i\times\mathbb{R}^J\mapsto\arg\max_{y\in B_i(\omega,z)}\ u_i(x,y)$ is a continuous mapping, since u_i and B_i are continuous. Hence, passing to the limit into relations (I) yields a continuous mapping, $\omega\in P_i\mapsto x_{i\omega}:=\arg\max_{y\in B_i(\omega,z_i^\eta)} u_i(x_{i0}^\eta,y)$, which, from Lemma 3-$(v)$ and above, is embedded into a continuous mapping, $x_i^\eta:\omega\in\{0\}\cup P_i^\eta\mapsto x_{i\omega}^\eta$, i.e., $x_i^\eta\in X(\pi_i^\eta)$. $\square$

Assertion (vii) Let $i\in I$ and $x_i^\eta\in X(\pi_i^\eta)$ be given, along Lemma 3-(vi). Let $\varphi_i:(x,\omega,z)\in\mathbb{R}^L_+\times P_i\times\mathbb{R}^J\mapsto\arg\max_{y\in B_i(\omega,z)}\ u_i(x,y)$ be defined on its domain. By the same token as for proving Assertion (vi), φ_i and $U_i : (x,\omega,z) \in \mathbb{R}^L_+ \times P_i \times \mathbb{R}^J \mapsto u_i(x,\varphi_i(x,\omega,z))$ are continuous mappings and, moreover, the relations $u_i(x^\eta_{i0}, x^\eta_{i\omega}) = U_i(x^\eta_{i0}, \omega, z^\eta_i)$ and $u_i(x^n_{i0}, x^n_{i\,\mathrm{arg}^n_i(\omega)}) = U_i(x^n_{i0}, \mathsf{arg}^n_i(\omega), z^n_i)$ hold, for every $(\omega, n) \in P_i \times \mathbb{N}_N$. Then, the uniform continuity of u_i and U_i on compact sets, and Lemma 3-(ii)-(iii)-(iv) yield:

(I) $\qquad \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N}_N : \forall n > N_\varepsilon,\ \forall \omega \in P_i,$

$$\mid u_i(x^\eta_{i0}, x^\eta_{i\omega}) - u_i(x^n_{i0}, x^n_{i\,\mathrm{arg}^n_i(\omega)}) \mid + \textstyle\sum_{s\in\underline{\mathbf{S}}} \mid u_i(x^\eta_{i0}, x^\eta_{i\omega^\eta_s}) - u_i(x^n_{i0}, x^n_{is}) \mid \ < \varepsilon.$$

Moreover, with $\beta := (1{+}\eta\#\underline{\mathbf{S}})$, we recall the following definitions, for every $n \in \mathbb{N}_N$:

(II) $\qquad \beta u_i^{\pi_i^\eta}(x^\eta_i) := \int_{\omega\in P_i} u_i(x^\eta_{i0}, x^\eta_{i\omega}) d\pi_i(\omega) + \eta \sum_{s\in\underline{\mathbf{S}}} u_i(x^\eta_{i0}, x^\eta_{is});$

(III) $\qquad u^n_i(x^n_i) := \int_{\omega\in P_i} u_i(x^n_{i0}, x^n_{i\,\mathrm{arg}^n_i(\omega)}) d\pi_i(\omega) + \eta \sum_{s\in\underline{\mathbf{S}}} u_i(x^n_{i0}, x^n_{is}).$

Then, Lemma 4-(vii) results immediately from relations (I)-(II)-(III) above. □

Proof of Lemma 4 It is similar to that of Lemma 3, hence, left to the reader. □

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